

5.3 (Book)
6.4 (Lay) The Gram-Schmidt Process

Consider again the set of vectors of exercise 3 (Section 6.3).

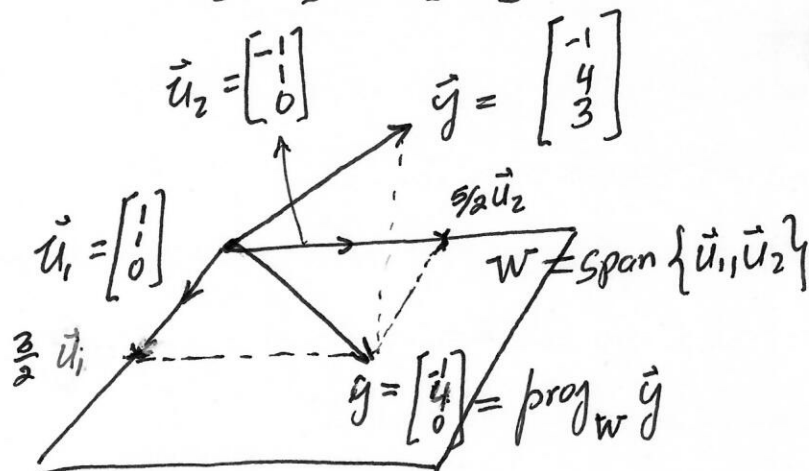
$$\vec{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

We found that $\vec{u}_1 \cdot \vec{u}_2 = 0$, $W = \text{span}\{u_1, u_2\}$

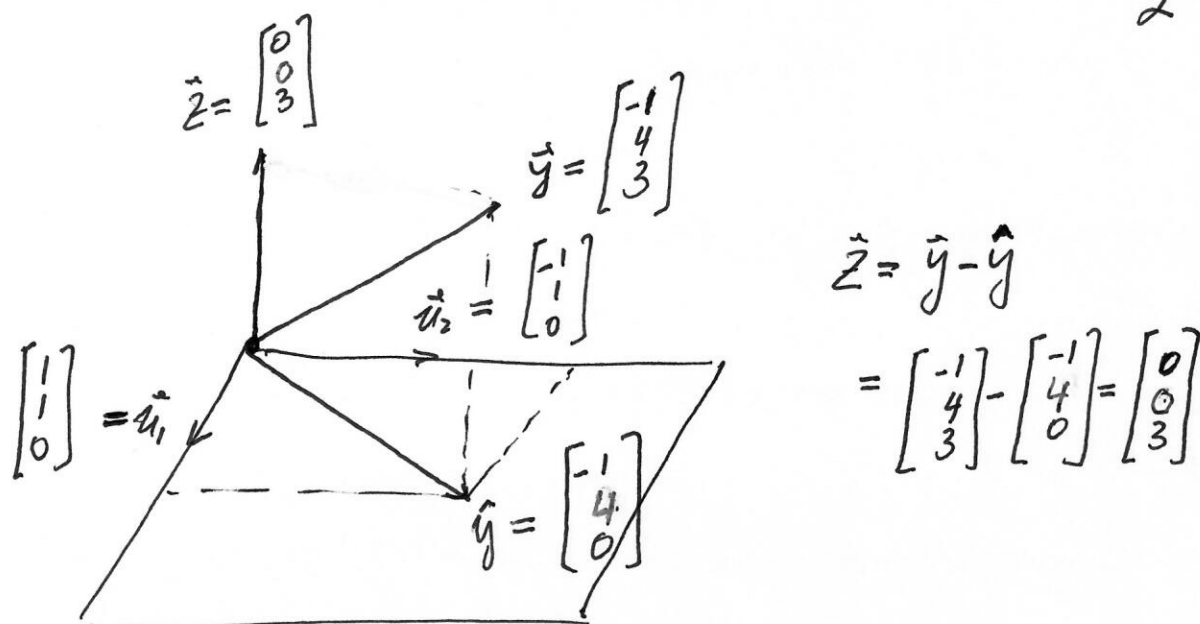
and $\text{proj}_W \vec{y} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$

We also found that $\vec{y}$ can be decomposed as

$$\vec{y} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} \in W + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \in W^\perp$$



Remark: Graphs in these notes are only for purposes of visualization. They do not reflect the precise positions and lengths of vectors.



i) The set $\{\vec{u}_1, \vec{u}_2, \vec{y}\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} \right\}$

form a basis for $\mathbb{R}^3$. (Prove it!). This basis is not orthogonal why?

ii) Construct an orthogonal basis for $\mathbb{R}^3$ from the vectors $u_1, u_2, \vec{y}$.

Ans:

$$i) \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, $\{\vec{u}_1, \vec{u}_2, \vec{y}\}$ is linearly independent.

ii) we already saw $\vec{u}_1 \cdot \vec{u}_2 = 0$ they are orthogonal

However, $\vec{u}_1 \cdot \vec{y} = -1 + 4 = 3 \neq 0$

$\vec{u}_2 \cdot \vec{y} = 1 + 4 = 5 \neq 0$

Therefore, the basis

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} \right\} \text{ is not orthogonal.}$$

Now, from our previous computation we found that

$$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \in W^\perp = (\text{Span}^W \{\vec{u}_1, \vec{u}_2\})^\perp.$$

Therefore, $\vec{z} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$ is orthogonal to $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and

$$\vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

and the set

$$B_0 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\} \text{ form an}$$

orthogonal basis for $\mathbb{R}^3$.

Notice that $\vec{z}$ was obtained from $\vec{u}_1, \vec{u}_2$ and $\vec{y}$ from the equation

$$\vec{z} = \vec{y} - \hat{y} = \vec{y} - \text{proj}_W \vec{y}$$

or

$$\boxed{\vec{z} = \vec{y} - \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 - \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2}$$

This suggests a procedure to obtain orthogonal bases $B_0 = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ from non-orthogonal basis $B = \{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$

Example

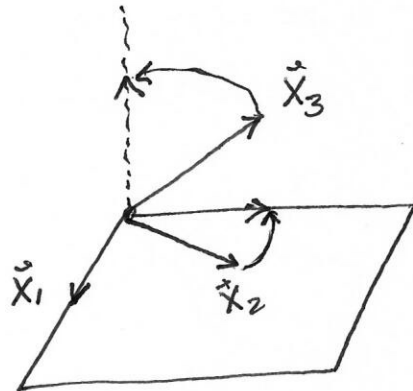
$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$$

i) Verify $B = \{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is not an orthogonal basis.

ii) Construct an orthogonal basis $B_0 = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ from $B = \{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$.

Ans: i) $\vec{x}_1 \cdot \vec{x}_2 \neq 0$, $\vec{x}_2 \cdot \vec{x}_3 \neq 0$, $\vec{x}_1 \cdot \vec{x}_3 \neq 0$

ii)



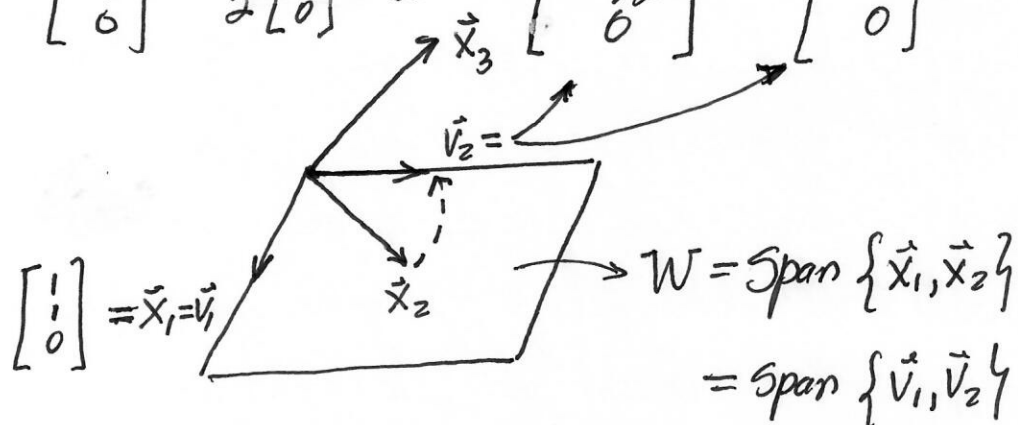
1st STEP: $\vec{v}_1 = \vec{x}_1$

2nd STEP: $\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{x}_1} \vec{x}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$

or

$$\vec{v}_2 = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} - \frac{[-2, 3, 0] \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{[1, 1, 0] \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 5/2 \\ 0 \end{bmatrix} \sim \begin{bmatrix} -5 \\ 5 \\ 0 \end{bmatrix}$$

STEP 3:

$$\vec{v}_3 = \vec{x}_3 - \text{proj}_W \vec{x}_3$$

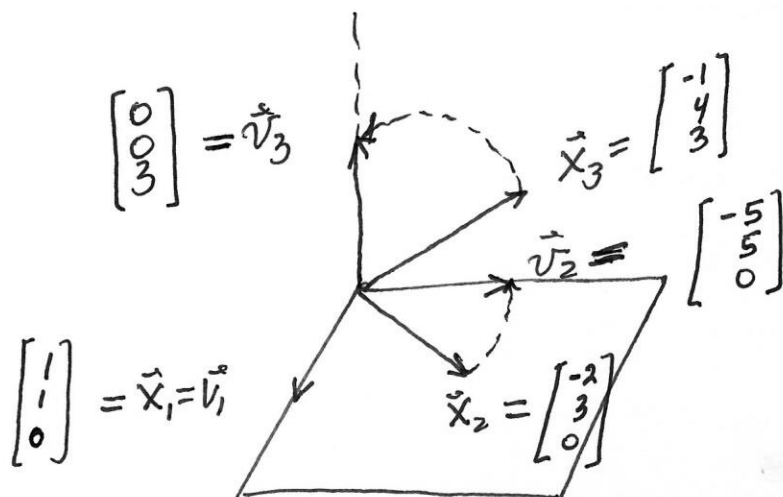
$$= \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$

$$= \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} - \frac{[-1, 4, 3] \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{[1, 1, 0] \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{[-1, 4, 3] \cdot \begin{bmatrix} -5 \\ 5 \\ 0 \end{bmatrix}}{[-5, 5, 0] \cdot \begin{bmatrix} -5 \\ 5 \\ 0 \end{bmatrix}} \begin{bmatrix} -5 \\ 5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{25}{50} \begin{bmatrix} -5 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 3/2 \\ 0 \end{bmatrix} - \begin{bmatrix} -5/2 \\ 5/2 \\ 0 \end{bmatrix}$$

or

$$\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$



Then, New $B_0 = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$

Notice

$$\text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} \right\} = \text{Span} \left\{ \begin{bmatrix} \vec{v}_1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \vec{v}_2 \\ -5 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} \vec{v}_3 \\ 0 \\ 0 \\ 3 \end{bmatrix} \right\}$$

The process applied is known as Gram-Schmidt process
And it is summarized in the following theorem:

Thm 11 A Given basis $B = \{\vec{x}_1, \dots, \vec{x}_p\}$ of a Subspace $W \subset \mathbb{R}^n$

($W \neq \{0\}$) Can be transformed into an orthogonal basis for W

$B_0 = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ by performing the following Computations:

$$\begin{aligned} \vec{v}_1 &= \vec{x}_1, & \vec{v}_2 &= \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1, & \vec{v}_3 &= \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 \\ &\vdots & & & & \vdots \\ \vec{v}_p &= \vec{x}_p - \frac{\vec{x}_p \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_p \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 - \dots - \frac{\vec{x}_p \cdot \vec{v}_{p-1}}{\vec{v}_{p-1} \cdot \vec{v}_{p-1}} \vec{v}_{p-1} \end{aligned}$$

In addition, $\text{Span}\{\vec{x}_1, \dots, \vec{x}_k\} = \text{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ for $1 \leq k \leq p$.

Section 6.4

Ex. 12 Find an orthogonal basis for the column space of the matrix

$$A = \begin{matrix} & \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \\ \begin{bmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 2 \\ 1 & 5 & 8 \end{bmatrix} \end{matrix}$$

First, we check if the set of columns of A is linearly independent

Row-reducing A

$$A \sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & 0 & 6 \\ 0 & 2 & 3 \\ 0 & 2 & -3 \\ 0 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & -3 \\ 0 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & -3 \\ 0 & 0 & 6 \\ 0 & 0 & 6 \\ 0 & 0 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & -3 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So the original vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3$ are linearly independent. and they form a basis for $\text{Col}(A)$.

Now, we apply the G-S process:

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{v}_1} \vec{x}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

$$\text{or } \vec{v}_2 = \begin{bmatrix} 3 \\ -3 \\ 2 \\ 5 \\ 5 \end{bmatrix} - \frac{16}{4} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad W = \text{Span}\{\vec{v}_1, \vec{v}_2\} = \text{Span}\{\vec{x}_1, \vec{x}_2\}$$

$$\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{v}_1} \vec{x}_2$$

$$\vec{v}_3 = \vec{x}_3 - \text{proj}_W \vec{x}_3$$

or

$$\vec{v}_3 = \vec{x}_3 - \text{proj}_{\vec{v}_1} \vec{x}_3 - \text{proj}_{\vec{v}_2} \vec{x}_3 =$$

$$= \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$

$$= \begin{bmatrix} 5 \\ 1 \\ 3 \\ 2 \\ 8 \end{bmatrix} - \frac{7}{4} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{8} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 1 \\ 3 \\ 2 \\ 8 \end{bmatrix} + \begin{bmatrix} -7/2 \\ +7/2 \\ 0 \\ -7/2 \\ -7/2 \end{bmatrix} + \begin{bmatrix} 3/2 \\ -3/2 \\ -3 \\ -3/2 \\ -3/2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \\ -3 \\ 3 \end{bmatrix}$$

Ex. 16

Find a QR factorization of the matrix A of Ex. 12.

We need to normalize the column vectors of matrix

$$B = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ 1 & -1 & 3 \\ -1 & 1 & 3 \\ 0 & 2 & 0 \\ 1 & 1 & -3 \\ 1 & 1 & 3 \end{bmatrix} \quad \text{to obtain } Q.$$

Notice that $\|\vec{v}_1\| = 2$, $\|\vec{v}_2\| = 2\sqrt{2}$, $\|\vec{v}_3\| = 6$

Then

$$Q = \begin{bmatrix} 1/2 & -1/2\sqrt{2} & 1/2 \\ -1/2 & 1/2\sqrt{2} & 1/2 \\ 0 & 1/\sqrt{2} & 0 \\ 1/2 & 1/2\sqrt{2} & -1/2 \\ 1/2 & 1/2\sqrt{2} & 1/2 \end{bmatrix}$$

We want obtain R Such that

$$A = QR$$

The columns of Q form an orthonormal basis for $\text{Col } A$. Then,

$$Q^T Q = I, \quad \text{why?}$$

Therefore, to find R Such that $A = QR$

is enough to multiply both sides by Q^T as follows:

$$Q^T A = (Q^T Q) R = I R = R.$$

In our case,

$$R = \begin{bmatrix} 1/2 & -1/2 & 0 & 1/2 & 1/2 \\ -1/2\sqrt{2} & 1/2\sqrt{2} & 1/\sqrt{2} & 1/2\sqrt{2} & 1/2\sqrt{2} \\ 1/2 & 1/2 & 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 2 \\ 1 & 5 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 8 & 7 \\ 0 & 2\sqrt{2} & 3\sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Upper triangular matrix
Invertible.

The factorization described in the previous exercise is of practical value for solving equations and finding eigenvalues. It's called "the QR Factorization".

Thm 12 $A_{m \times n}$ has n linearly independent columns.
Then

$$A = QR$$

where $Q_{m \times n}$ matrix: whose columns form an orthonormal basis for $\text{Col}(A)$

and $R_{n \times n}$ matrix: upper triangular and invertible.
with positive diagonal entries.